Exercise-sheet 8 (December 7, 2017)

1 In-class exercises

1.1 Self-consistent Hartree-Fock approximation

Consider the generic Hamiltonian $H = H_0 + H_1$, where

$$H_0 = \int d^3x\, \psi^\dagger(\boldsymbol{x}) \left(-\frac{\nabla^2}{2m} - \mu + U(\boldsymbol{x}) \right) \psi(\boldsymbol{x}), \tag{1}$$

with $U(\boldsymbol{x})$ a static spin-independent external potential and

$$H_1 = \frac{1}{2} \iint d^3x d^3x' \psi^\dagger(\boldsymbol{x}) \psi^\dagger(\boldsymbol{x}') V(\boldsymbol{x} - \boldsymbol{x}') \psi(\boldsymbol{x}') \psi(\boldsymbol{x}), \tag{2}$$

where $V(\boldsymbol{x} - \boldsymbol{x}')$ is a two-body potential.

(a) Draw (the diagrams of) the self-consistent Hartree-Fock approximation for the proper self-energy and the corresponding Dyson's equation, in real space.

(b) Given the following expansion for $\tilde{G}(\boldsymbol{x}, \boldsymbol{x}', \omega_n)$ and $\tilde{G}^{(0)}(\boldsymbol{x}, \boldsymbol{x}', \omega_n)$

$$\tilde{G}(\boldsymbol{x}, \boldsymbol{x}', \omega_n) = \sum_j \frac{\phi_j(\boldsymbol{x}) \phi_j^*(\boldsymbol{x}')}{i\omega_n - (\epsilon_j - \mu)} \quad \text{and} \quad \tilde{G}^{(0)}(\boldsymbol{x}, \boldsymbol{x}', \omega_n) = \sum_j \frac{\phi_j^{(0)}(\boldsymbol{x}) \phi_j^{(0)*}(\boldsymbol{x}')}{i\omega_n - (\epsilon_j^{(0)} - \mu)}, \tag{3}$$

with $H\phi_j(\boldsymbol{x}) = (\epsilon_j - \mu)\phi_j(\boldsymbol{x})$ and $H_0 \phi_j^{(0)}(\boldsymbol{x}) = (\epsilon_j^{(0)} - \mu)\phi_j^{(0)}(\boldsymbol{x})$, find the self-consistent Hartree-Fock equations for the set $\{\phi_j(\boldsymbol{x})\}$.

2 Homework - due date: December 15, 2017 (30 points).

2.1 Yet another pair of frequency summations (10 points)

Show

$$-\frac{1}{\beta} \sum_{\omega_n} \tilde{G}^{(0)}(\boldsymbol{p}, \omega_n) \tilde{G}^{(0)}(\boldsymbol{k}, \nu_m - \omega_n) = \frac{1 - n_F(\epsilon_{\boldsymbol{p}}) - n_F(\epsilon_{\boldsymbol{k}})}{i\nu_m - \epsilon_{\boldsymbol{p}} - \epsilon_{\boldsymbol{k}}}, \tag{4}$$

and

$$\lim_{\tau \to 0^-} G^{(0)}(\tau, 0) = n_F(\epsilon_{\boldsymbol{k}}). \tag{5}$$

Here $n_F(\epsilon_{\boldsymbol{p}})$ is the Fermi function and ω_n (ν_m) are fermionic (bosonic) frequencies.

Note that in equation (5) $G^{(0)}(\tau) = \dfrac{1}{\beta}\sum_{\omega_n} e^{-i\omega_n \tau}\tilde{G}^{(0)}(\omega_n)$, as usual.

(Hint: To prove equation (5), remember that the complex function $G(z)$ has a branch cut on the real line. You will also need to use the definition of the spectral function for non-interacting electrons).

2.2 First order perturbation theory (10 points)

Consider a system of electrons interacting through a generic spin- and time-independent two-body interaction. Furthermore, consider the following integrals of time-ordered products of field operators, which appear (divided by the partition function) as the first order contribution in the perturbative expansion for the interacting Green's function $G(\boldsymbol{x}\tau, \boldsymbol{x'}\tau')$ of a particle propagating from $(\boldsymbol{x'}, \tau')$ to $(\boldsymbol{x}, \tau)$

$$\frac{1}{2}\iint_0^\beta d\tau_1 d\tau_2 \iint d^3x_1 d^3x_2 V(\boldsymbol{x}_1 - \boldsymbol{x}_2)\delta(\tau_1 - \tau_2)\langle T_\tau \psi(\boldsymbol{x}\tau)\psi(\boldsymbol{x}_1\tau_1)\psi(\boldsymbol{x}_2\tau_2)\psi^\dagger(\boldsymbol{x}_2\tau_2)\psi^\dagger(\boldsymbol{x}_1\tau_1)\psi^\dagger(\boldsymbol{x'}\tau')\rangle, \tag{6}$$

where $\langle T_\tau \cdot \rangle = \mathrm{Tr}\{e^{-\beta \hat{H}_0} T_\tau(\cdot)\}$ and $V(\boldsymbol{x})$ is the spin-independent, static two-body interaction.

(a) Use Wick's theorem to express the expectation value in (6) as a sum of products of non-interacting Green's functions in real space.

(b) After using Wick's theorem you are left with a sum of integrals over non-interaction Green's functions and two-body interaction terms. Assign a diagram to each of the resulting integrals.

2.3 First order perturbation theory, part II (10 points)

(a) Consider again a system of electrons interacting through a generic spin- and time-independent two-body interaction $V(\boldsymbol{k})$. Use Feynman rules in momentum space to draw all **connected** diagrams which contribute to the interacting imaginary-time Green's function $\tilde{G}(\boldsymbol{k}, \omega_n)$, **to zero and first order** in V.

(b) Evaluate these diagrams (i.e. write down the analytic expression for each diagram using Feynman rules and do the corresponding frequency summations) and give the resulting expression for $\tilde{G}(\boldsymbol{k}, \omega_n)$ in terms of the two-body potential $V(\boldsymbol{k})$.