Exercise-sheet 7 (December 1, 2017)

1 Homework - due date: December 8, 2017 (20 points).

1.1 Frequency summations I (10 points)

Let's consider the following sum

$$S = -\frac{1}{\beta}\sum_{\nu_n} \tilde{D}^{(0)}(\boldsymbol{q},\nu_n)\,\tilde{G}^{(0)}(\boldsymbol{k},\omega_n+\nu_n), \tag{1}$$

with ν_n (ω_n) bosonic (fermionic) Matsubara frequencies and

$$\tilde{G}^{(0)}(\boldsymbol{k},\xi_n) = \frac{1}{i\xi_n - \epsilon_{\boldsymbol{k}}}, \quad \tilde{D}^{(0)}(\boldsymbol{q},\xi_n) = \frac{2\,\omega_{\boldsymbol{q}}}{(i\xi_n)^2 - \omega_{\boldsymbol{q}}^2}, \tag{2}$$

the free-electron Green's function and the free-phonon Green's function.

Transform the infinite sum into a contour integral using the Residue theorem and the fact that the Bose function

$$n_B(\omega_{\boldsymbol{q}}) = \frac{1}{e^{\beta\omega_{\boldsymbol{q}}}-1} = -\frac{1}{2} + \frac{1}{\beta\omega_{\boldsymbol{q}}} + \frac{1}{\beta}\sum_{n=1}^{\infty}\left(\frac{1}{\omega_{\boldsymbol{q}}+i\nu_n} + \frac{1}{\omega_{\boldsymbol{q}}-i\nu_n}\right), \tag{3}$$

with $\nu_n = \frac{2\pi n}{\beta}$. Deform the contour and calculate the sum.

1.2 Frequency summations II (10 points)

Show

$$\frac{1}{\beta}\sum_{\omega_n} \tilde{G}^{(0)}(\boldsymbol{p},\omega_n)\,\tilde{G}^{(0)}(\boldsymbol{k},\omega_n+\nu_n) = \frac{n_F(\epsilon_{\boldsymbol{p}}) - n_F(\epsilon_{\boldsymbol{k}})}{i\nu_n + \epsilon_{\boldsymbol{p}} - \epsilon_{\boldsymbol{k}}}, \tag{4}$$

where $n_F(\epsilon_{\boldsymbol{p}})$ is the Fermi function and ω_n (ν_n) are fermionic (bosonic) frequencies, using the fact that

$$n_F(\epsilon_{\boldsymbol{p}}) = \frac{1}{2} - \frac{1}{\beta}\sum_{n=0}^{\infty}\left(\frac{1}{\epsilon_{\boldsymbol{p}}+i\omega_n} + \frac{1}{\epsilon_{\boldsymbol{p}}-i\omega_n}\right), \quad \text{with} \quad \omega_n = \frac{(2n+1)\pi}{\beta}. \tag{5}$$