# Exercise-sheet 10 (January 12, 2018)

## 1 In-class exercises

In this session we will mainly revisit some of the concepts and assumptions behind Landau's Fermi-liquid theory

### 1.1 Quasi-particle distribution in Landau's theory

In a macroscopic state of thermal equilibrium, the smoothed distribution function $n_{\boldsymbol{p}\sigma}$ may be determined from the fact that for *any* variation about thermodynamic equilibrium at finite temperature

$$\delta E = T\delta s + \mu\delta n, \tag{1}$$

where $E$ is the energy (per unit volume) of the system, $\delta s$ is the variation of the entropy density, $\delta n$ is the variation of the particle density, $T$ is the temperature and $\mu$ is the chemical potential.

(I) Show $n_{\boldsymbol{p}\sigma}$ has the functional form

$$n_{\boldsymbol{p}\sigma} = \frac{1}{e^{\beta(\epsilon_{\boldsymbol{p}\sigma}-\mu)}+1}, \tag{2}$$

where $\epsilon_{\boldsymbol{p}\sigma} = \epsilon_{\boldsymbol{p}\sigma}[n_{\boldsymbol{p}\sigma}]$ is the quasi-particle energy.

(II) Use the last expression to show the zero-temperature density of quasi-particles at the Fermi surface is

$$N(0) = \frac{m^* p_f}{\pi^2}, \tag{3}$$

where $m^*$ is the effective mass defined as $p_f = m^* v_f$.

In the exercises below you shall see that thermodynamic quantities at low temperatures depend on $N(0)$.

### 1.2 Quasi-particle current and backflow in Landau's Fermi-liquid theory

(a) Show the current $J_{\boldsymbol{k}\alpha} = -\partial\varepsilon_{\boldsymbol{k}}/\partial q_\alpha$ (where does this expression come from?) carried by a quasi-particle $\boldsymbol{k}$ is given by

$$J_{\boldsymbol{k}\alpha} = v_{\boldsymbol{k}\alpha} - \sum_{\boldsymbol{k}'} f_{\boldsymbol{k}\boldsymbol{k}'} \frac{\delta n_{\boldsymbol{k}'}}{q_\alpha}, \tag{4}$$

where $\partial\varepsilon_{\boldsymbol{k}}/\partial q_\alpha$ expresses the variation in the energy $\varepsilon_{\boldsymbol{k}}$, when the coordinate system is displaced with the velocity $\boldsymbol{q}/m$ (or when all the particles are displaced with the velocity $-\boldsymbol{q}/m$).

(b) Compute $\delta n_{\boldsymbol{k}}$ for a fixed direction of $\boldsymbol{k}$ making an angle $\theta$ with $\boldsymbol{q}$, and show eq. (4) simplifies to

$$J_{\boldsymbol{k}\alpha} = v_{\boldsymbol{k}\alpha} + \frac{V}{(2\pi)^3} \sum_{\sigma'} \int d^3k' v_{\boldsymbol{k}'\alpha} \delta(\varepsilon_{\boldsymbol{k}'} - \mu) f_{\boldsymbol{k}\boldsymbol{k}'}. \tag{5}$$

## 2 Homework - due date: January 19, 2018 (30 points).

### 2.1 Entropy and specific heat in Landau's Fermi-liquid theory (25 points)

(I) Use

$$n = \frac{1}{V} \sum_{\boldsymbol{p}\sigma} n_{\boldsymbol{p}\sigma} \quad \text{and} \quad s = -\frac{1}{V} \sum_{\boldsymbol{p}\sigma} [n_{\boldsymbol{p}\sigma} \ln n_{\boldsymbol{p}\sigma} + (1 - n_{\boldsymbol{p}\sigma}) \ln(1 - n_{\boldsymbol{p}\sigma})], \tag{6}$$

together with eq. (2) to show that under variations of the temperature

$$\delta s = \frac{1}{TV} \sum_{\boldsymbol{p}\sigma} (\epsilon_{\boldsymbol{p}\sigma} - \mu) \delta n_{\boldsymbol{p}\sigma}, \tag{7}$$

with

$$\delta n_{\boldsymbol{p}\sigma} = \frac{\partial n_{\boldsymbol{p}\sigma}}{\partial \epsilon_{\boldsymbol{p}\sigma}} \left( -\frac{\epsilon_{\boldsymbol{p}\sigma} - \mu}{T} \delta T + \delta\epsilon_{\boldsymbol{p}\sigma} - \delta\mu \right), \tag{8}$$

where the term $(\delta\epsilon_{\boldsymbol{p}\sigma} - \delta\mu) \sim T^3 \log T$ and can be neglected. It thus follows the first contribution to the entropy is due to the explicit $\delta T$ and is

$$\delta s = -\frac{1}{V} \sum_{\boldsymbol{p}\sigma} \frac{\partial n_{\boldsymbol{p}\sigma}}{\partial \epsilon_{\boldsymbol{p}\sigma}} (\epsilon_{\boldsymbol{p}\sigma} - \mu)^2 \frac{\delta T}{T^2}. \tag{9}$$

(II) Replace the sum by an integral over energy and show that

$$s = \frac{\pi^2}{3} N(0) T, \tag{10}$$

and that the specific heat

$$c_V = T \left( \frac{\partial s}{\partial T} \right)_V = s = \frac{m^* p_f}{3} T, \tag{11}$$

## 2.2 Galilean invariance in isotropic liquids (5 points)

Use eq. (5) to show that for a translationally invariant system the following identity holds

$$\frac{1}{m} = \frac{1}{m^*} + \frac{V}{(2\pi)^3} k_F \sum_{\sigma'} \int d\Omega f(\sigma, \sigma', \theta) \cos\theta. \tag{12}$$