Exercise-sheet 8 (December 14, 2015)

1 Feynman rules...

1.1 In coordinate space

(1) Draw all topologically distinct diagrams containing n interaction lines and $2n + 1$ directed particle lines.

(2) Associate a factor $G^{(0)}(\boldsymbol{x}_1, \tau_1, \boldsymbol{x}_2, \tau_2)$ with each directed particle line running from $(\boldsymbol{x}_2, \tau_2)$ to $(\boldsymbol{x}_1, \tau_1)$.

(3) Associate a factor $V(\boldsymbol{x}_1 - \boldsymbol{x}_2)\delta(\tau_2 - \tau_1)$ with each interaction line joining $(\boldsymbol{x}_1, \tau_1)$ and $(\boldsymbol{x}_2, \tau_2)$.

(4) Integrate all internal variables x_i, τ_i.

(5) Multiply each nth order diagram by $(-1)^{n+F}$, where F is the number of closed fermion loops.

(6) Interpret any temperature Green's function at equal values of τ as

$$G^{(0)}(\boldsymbol{x}_i, \tau_i, \boldsymbol{x}_j, \tau_i) = \lim_{\tau_j \to \tau_i^+} G^{(0)}(\boldsymbol{x}_i, \tau_i, \boldsymbol{x}_j, \tau_j)$$

1.2 In momentum space

(1) Draw all topologically distinct diagrams containing n interaction lines and $2n + 1$ directed particle lines.

(2) Assign a direction to each interaction line.

(3) Associate a wave vector and discrete (Matsubara) frequency with each line, and conserve each quantity at every vertex.

(4) With each particle line associate a factor

$$\tilde{G}^{(0)}(\boldsymbol{k}, \omega_n) = \frac{1}{i\omega_n - (\epsilon_{\boldsymbol{k}} - \mu)},$$

where ω_n are the Matsubara frequencies.

(5) Associate a factor $V(\boldsymbol{k})$ with each interacting line.

(6) Integrate over all n independent internal wave vectors and sum over all n independent internal frequencies.

(7) Multiply by $(-8\pi^3\beta)^{-n}(-1)^F$, where F is the number of closed fermion loops.

(8) Whenever a particle line either closes on itself or is joined by the same interaction line, insert a convergence factor $e^{i\omega_n\delta}$

2 Homework - due date: December 21, 2015 (25 points).

2.1 First order perturbation theory (15 points)

Consider a system of electrons interacting through a generic spin- and time-independent two-body interaction. Furthermore, consider the following integrals of time-ordered products of field operators, which appear (divided by the partition function) as the first order contribution in the perturbative expansion for the interacting Green's function $G(\boldsymbol{x}\tau, \boldsymbol{x'}\tau')$ of a particle propagating from $(\boldsymbol{x'}, \tau')$ to $(\boldsymbol{x}, \tau)$

$$\frac{1}{2}\iint_0^\beta d\tau_1 d\tau_2 \iint d^3x_1 d^3x_2 V(\boldsymbol{x}_1-\boldsymbol{x}_2)\delta(\tau_1-\tau_2)\langle T_\tau \psi(\boldsymbol{x}\tau)\psi(\boldsymbol{x}_1\tau_1)\psi(\boldsymbol{x}_2\tau_2)\psi^\dagger(\boldsymbol{x}_2\tau_2)\psi^\dagger(\boldsymbol{x}_1\tau_1)\psi^\dagger(\boldsymbol{x'}\tau')\rangle, \tag{1}$$

where $\langle T_\tau \cdot\rangle = \mathrm{Tr}\{e^{-\beta \hat{H}_0} T_\tau(\cdot)\}$ and $V(\boldsymbol{x})$ is the spin-independent, static two-body interaction.

(a) Use Wick's theorem to express the expectation value in (1) as a sum of products of non-interacting Green's functions in real space.

(b) After using Wick's theorem you are left with a sum of integrals over non-interaction Green's functions and two-body interaction terms. Assign a diagram to each of the resulting integrals.

2.2 First order perturbation theory, part II (10 points)

(a) Consider again a system of electrons interacting through a generic spin- and time-independent two-body interaction $V(\boldsymbol{k})$. Use Feynman rules in momentum space to draw all **connected** diagrams which contribute to the interacting imaginary-time Green's function $\tilde{G}(\boldsymbol{k}, \omega_n)$, **to zero and first order** in V.

(b) Evaluate these diagrams (i.e. write down the analytic expression for each diagram using Feynman rules and do the corresponding frequency summations) and give the resulting expression for $\tilde{G}(\boldsymbol{k}, \omega_n)$ in terms of the two-body potential $V(\boldsymbol{k})$.