Exercise-sheet 7 (December 7, 2015)

1 In-class exercises

1.1 More on frequency summation

Prove

$$-\frac{1}{\beta}\sum_{\boldsymbol{k}}\sum_{n} e^{-i\omega_n 0^+}\log(\epsilon_{\boldsymbol{k}} - i\omega_n) = F, \tag{1}$$

with F the free-energy for non-interacting electrons.

1.2 Spectral function of non-interacting electrons and quasi-particles (revisited)

(a) Calculate the spectral function for a system of non-interacting electrons.

(b) How does the spectral function change, if the single-electron Green's function decays exponentially in time with some characteristic time scale?

2 Homework - due date: December 14, 2015 (30 + 5 points).

2.1 Greens function of an impurity immersed in a fermionic bath (5 points)

Show the complex function

$$G(z) = \frac{1}{z - \epsilon - \displaystyle\sum_{\boldsymbol{k}} \frac{V^2}{z - \epsilon_{\boldsymbol{k}}}}, \tag{2}$$

is the correct analytic continuation of $\tilde{G}(\omega_n)$ (which was calculated in homework 5), by

(a) Proving $G(z)$ is analytic in the upper half-plane (i.e, show the denominator does not vanish if $\operatorname{Im} z$ is not zero).

(b) Showing $\lim\limits_{|z|\to\infty} z\, G(z) = \text{constant}$.

2.2 Yet another pair of frequency summations (10 points)

Show

$$-\frac{1}{\beta}\sum_{\omega_n}\tilde{G}^{(0)}(\boldsymbol{p},\omega_n)\tilde{G}^{(0)}(\boldsymbol{k},\nu_m-\omega_n)=\frac{1-n_F(\epsilon_{\boldsymbol{p}})-n_F(\epsilon_{\boldsymbol{k}})}{i\nu_m-\epsilon_{\boldsymbol{p}}-\epsilon_{\boldsymbol{k}}}, \tag{3}$$

and

$$\lim_{\tau\to 0^-} G^{(0)}(\tau,0)=n_F(\epsilon_{\boldsymbol{k}}). \tag{4}$$

Here $n_F(\epsilon_{\boldsymbol{p}})$ is the Fermi function and ω_n (ν_m) are fermionic (bosonic) frequencies.

Note that in equation (4) $G^{(0)}(\tau)=\frac{1}{\beta}\sum_{\omega_n} e^{-i\omega_n}\tilde{G}^{(0)}(\omega_n)$, as usual.

(Hint: To prove equation (4), remember that the complex function $G(z)$ has a branch cut on the real line. You will also need to use the definition of the spectral function for non-interacting electrons).

2.3 Wick's theorem for non-operators (15 points)

In the lecture we studied the single-electron Green's function $G(\boldsymbol{k})$ in a many-impurity system. We are interested in the impurity average of the Green's function, to calculate it we need to evaluate averaged products of the form

$$\langle \rho_{\boldsymbol{q_1}}\rho_{\boldsymbol{q_2}}\cdots\rho_{\boldsymbol{q_n}}\rangle, \tag{5}$$

where $\langle\cdot\rangle$ is not a thermodynamic average but an ensemble average, $\rho_{\boldsymbol{q}}=\sum_i e^{-i\boldsymbol{q}\cdot\boldsymbol{R}_i}$ is the Fourier transform of particle density of impurities and $\boldsymbol{R}_i$ is the location of the i-th impurity.

(a) Use the definition of ensemble average given in the lecture to evaluate $\langle\rho_{\boldsymbol{q_1}}\rangle$, $\langle\rho_{\boldsymbol{q_1}}\rho_{\boldsymbol{q_2}}\rangle$ and $\langle\rho_{\boldsymbol{q_1}}\rho_{\boldsymbol{q_2}}\rho_{\boldsymbol{q_3}}\rangle$.

(b) Draw the Feynman diagrams for each contribution from $\langle\rho_{\boldsymbol{q_1}}\rho_{\boldsymbol{q_2}}\rho_{\boldsymbol{q_3}}\rangle$.

(c) (BONUS + 5 points) Gives an expression for the n-th order average $\langle\rho_{\boldsymbol{q_1}}\rho_{\boldsymbol{q_2}}\cdots\rho_{\boldsymbol{q_n}}\rangle$