Exercise-sheet 11 (January 18, 2015)

1 In-class exercises

1.1 Quasi-particle current and backflow in Landau's Fermi-liquid theory

(a) *How should we define the current of a quasi-particle?*

In an arbitrary state $|\varphi\rangle$, the current $\boldsymbol{J}$ is given by

$$\boldsymbol{J} = \langle\varphi|\sum_i \boldsymbol{p}_i/m|\varphi\rangle, \tag{1}$$

where $\boldsymbol{p}_i$ is the momentum of the ith particle of a system described by the Hamiltonian

$$H = \sum_i \frac{p_i^2}{2m} + W, \tag{2}$$

where the interaction W depends only on the position and the relative velocities of the particles, i.e. it's not modified by a translation.

Show that when q tends to zero

$$J_\alpha = -\frac{\partial E_{\boldsymbol{q}}}{\partial q_\alpha}, \tag{3}$$

with $E_{\boldsymbol{q}}$ the expectation value of H in the state $|\phi\rangle$, when measured from a frame which moves with respect to the system with a uniform velocity $\boldsymbol{q}/m$.

(b) Show the current $J_{\boldsymbol{k}\alpha} = -\partial\varepsilon_{\boldsymbol{k}}/\partial q_\alpha$ carried by a quasi-particle $\boldsymbol{k}$ is given by

$$J_{\boldsymbol{k}\alpha} = v_{\boldsymbol{k}\alpha} - \sum_{\boldsymbol{k'}} f_{\boldsymbol{k}\boldsymbol{k'}} \frac{\delta n_{\boldsymbol{k'}}}{q_\alpha}, \tag{4}$$

where $\partial\varepsilon_{\boldsymbol{k}}/\partial q_\alpha$ expresses the variation in the energy $\varepsilon_{\boldsymbol{k}}$, when the coordinate system is displaced with the velocity $\boldsymbol{q}/m$ (or when all the particles are displaced with the velocity $-\boldsymbol{q}/m$).

(c) Compute $\delta n_{\boldsymbol{k}}$ for a fixed direction of $\boldsymbol{k}$ making an angle θ with $\boldsymbol{q}$, and show eq. 4 simplifies to

$$J_{\boldsymbol{k}\alpha} = v_{\boldsymbol{k}\alpha} + \frac{V}{(2\pi)^3}\sum_{\sigma'}\int d^3k' v_{\boldsymbol{k'}\alpha}\delta(\varepsilon_{\boldsymbol{k'}} - \mu) f_{\boldsymbol{k}\boldsymbol{k'}}. \tag{5}$$

(d) Show that for a translationally invariant system the following identity holds

$$\frac{1}{m} = \frac{1}{m^*} + \frac{V}{(2\pi)^3} k_F \sum_{\sigma'} \int d\Omega f(\sigma, \sigma', \theta) \cos\theta. \tag{6}$$

1.2 Boltzmann equation for quasi-particles and conservation laws

The Boltzmann equation for the particle distribution function $n = n(\boldsymbol{k}, \boldsymbol{r}, t)$ is given by

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial r_\alpha}\frac{\partial r_\alpha}{\partial t} + \frac{\partial n}{\partial k_\alpha}\frac{\partial k_\alpha}{\partial t} = I(n), \tag{7}$$

where $I(n)$ is the so-called collision integral.

A quasi-particle $\boldsymbol{k}$ located at the point $\boldsymbol{r}$ has an energy

$$\varepsilon(\boldsymbol{k}, \boldsymbol{r}) = \varepsilon_{\boldsymbol{k}}^{(0)} + \sum_{\boldsymbol{k'}} f_{\boldsymbol{k}\boldsymbol{k'}} \delta n(\boldsymbol{k'}, \boldsymbol{r}), \tag{8}$$

and hence the quasi-particles feel a diffusive force $F_\alpha = -\partial\varepsilon/\partial r_\alpha$ which tends to push them towards regions of minimum energy.

(a) Show that to first order in $\delta n(\boldsymbol{k}, \boldsymbol{r}, t) = n(\boldsymbol{k}, \boldsymbol{r}, t) - n_{\boldsymbol{k}}^{(0)}$ and in the absence of any external field, the (linearized) Boltzmann equation takes the form

$$\frac{\partial \delta n(\boldsymbol{k}, \boldsymbol{r})}{\partial t} + v_{\boldsymbol{k}\alpha}\frac{\partial \delta n}{\partial r_\alpha} + v_{\boldsymbol{k}\alpha}\delta(\varepsilon_{\boldsymbol{k}} - \mu) \sum_{\boldsymbol{k'}} f_{\boldsymbol{k}\boldsymbol{k'}} \frac{\partial \delta n(\boldsymbol{k'}, \boldsymbol{r})}{\partial r_\alpha} = I(n) \tag{9}$$

(b) Define $\delta n(\boldsymbol{r}) = \sum_{\boldsymbol{k}} \delta n(\boldsymbol{k}, \boldsymbol{r})$ and sum eq. (9) over all values of $\boldsymbol{k}$ to obtain

$$\frac{\partial \delta n}{\partial t} + \mathrm{div}\boldsymbol{J} = 0. \tag{10}$$

2 Homework - due date: January 25, 2016 (25 points).

2.1 Thermodynamic stability of Landau's Fermi-liquid theory

In Landau's theory we assume the distribution function of the ground state of an interacting system is the same as that of an ideal gas (i.e. an isotropic step function). This assertion is valid only if the state so defined is stable, corresponding to a minimum of the Free energy. This leads us to study the stability of the Fermi surface.

Let's define a direction in $\boldsymbol{k}$-space by the spherical angles θ and φ, and let's displace the Fermi-surface of spin σ in this direction by an infinitesimal amount $u = u(\theta, \varphi, \sigma)$.

Under this distortion the Free energy varies as

$$\delta F = \sum_{\boldsymbol{k}} (\varepsilon_{\boldsymbol{k}} - \mu)\delta n_{\boldsymbol{k}} + \frac{1}{2}\sum_{\boldsymbol{k}\boldsymbol{k}'} f_{\boldsymbol{k}\boldsymbol{k}'}\delta n_{\boldsymbol{k}}\delta n_{\boldsymbol{k}'}. \tag{11}$$

(a) Use $\delta n_{\boldsymbol{k}} = 1$ if $k_F < |\boldsymbol{k}| < k_F + u$ and $\delta n_{\boldsymbol{k}} = 0$ otherwise to show that (to lowest order in u)

$$\delta F = \frac{V}{(2\pi)^3}\frac{k_F^3}{2m^*}\sum_{\sigma}\int d\Omega u^2(\theta,\varphi,\sigma) + \frac{V^2}{(2\pi)^6}\frac{k_F^4}{2}\sum_{\sigma\sigma'}\int d\Omega u(\theta,\varphi,\sigma)\int d\Omega' u(\theta',\varphi',\sigma') f(\xi,\sigma,\sigma'), \tag{12}$$

where ξ is the angle between the directions (θ,φ) and (θ',φ').

(b) Assume u and f are spin-independent and decompose them into spherical harmonics as

$$u = \sum_{lm} u_{lm} Y_{lm}(\theta,\varphi) \tag{13}$$

$$f(\xi) = \sum_{l} f_l P_l(\cos\xi), \tag{14}$$

with $f_l = \frac{2l+1}{4\pi}\int f(\xi)P_l(\cos\xi)d\Omega$, to obtain

$$\delta F = \frac{Vk_F^3}{(2\pi)^3}\frac{1}{m^*}\sum_{lm}|u_{lm}|^2\frac{4\pi}{2l+1}\frac{(l+m)!}{(l-m)!}\left(1 + \frac{m^*}{2l+1}\frac{Vk_F f_l}{\pi^2}\right). \tag{15}$$

(c) What are then the conditions for thermodynamic stability in Landau's theory?

2.2 Response of an electron gas to an electric field

Let $E^{ext} = E^{ext}(\boldsymbol{r},t)$ be an external electric field applied to a system of interacting electrons. Let's assume the field is periodic in space and time, having the form $E^{ext}(\boldsymbol{r},t) = E^{ext}e^{i(\boldsymbol{q}\cdot\boldsymbol{r}-\omega t)}$. This field induces currents in the gas which in turn generate other electric fields. The summation of all these fields yields the local field $E(\boldsymbol{r},t) = Ee^{i(\boldsymbol{q}\cdot\boldsymbol{r}-\omega t)}$ felt by the particles. The deformation δn due to this local field varies according to the same law, i.e. $\delta n(\boldsymbol{k},\boldsymbol{r},t) = \delta n_{\boldsymbol{k}}e^{i(\boldsymbol{q}\cdot\boldsymbol{r}-\omega t)}$.

For Landau's theory to be applicable, $q \ll k_F$ and $\omega \ll \mu$ (why is this so?). Furthermore, we assume ω is much greater than the collision frequency, so that we can neglect the collision integral.

(a) Use $n(\boldsymbol{k},\boldsymbol{r},t) = n_{\boldsymbol{k}}^{(0)} + \delta n(\boldsymbol{k},\boldsymbol{r},t)$ to obtain the linearized Boltzmann equation

$$(\boldsymbol{q}\cdot\boldsymbol{v}_{\boldsymbol{k}} - \omega)\delta n(\boldsymbol{k}) + \boldsymbol{q}\cdot\boldsymbol{v}_{\boldsymbol{k}}\delta(\varepsilon_{\boldsymbol{k}} - \mu)\sum_{\boldsymbol{k}'} f_{\boldsymbol{k}\boldsymbol{k}'}\delta n_{\boldsymbol{k}'} + ie\boldsymbol{E}\cdot v_{\boldsymbol{k}}\delta(\varepsilon_{\boldsymbol{k}} - \mu) = 0 \tag{16}$$

(b) The electric current density $I(\boldsymbol{r}, t)$ is given by

$$\boldsymbol{I}(\boldsymbol{r}, t) = e \sum_{\boldsymbol{k}} \delta n_{\boldsymbol{k}} \boldsymbol{J}_{\boldsymbol{k}} e^{i(\boldsymbol{q}\cdot\boldsymbol{r} - \omega t)}. \tag{17}$$

Within linear response $I_\alpha = \sigma_{\alpha\beta} E_\beta$, where $\sigma_{\alpha\beta}(\boldsymbol{q}, \omega)$ is the conductivity tensor. Use Boltzmann's equation to calculate $\sigma_{\alpha\beta}(0, \omega)$ for a translationally invariant system.