Exercise-sheet 6 (June 20, 2017)

1 Homework - due Date: June 27, 2017 (30 points).

1.1 Density of states for gapped systems (10 points)

Within BCS theory the excitation spectrum of the single electron state is given by

$$E(\boldsymbol{k}) = \sqrt{\xi(\boldsymbol{k})^2 + \Delta(\boldsymbol{k})^2}, \tag{1}$$

where $\xi(\boldsymbol{k})$ is measured from the Fermi energy E_F.

Assuming $\Delta(\boldsymbol{k}) = \Delta$ and $\Delta \ll E_F$, compute the density of states.

1.2 Temperature dependence of the gap function (20 points)

(a) In the weak-coupling limit ($\Delta_0 \ll \hbar\omega_D$), show that the BCS gap equation

$$1 = gN(0) \int_0^{\hbar\omega_D} \frac{d\xi}{(\xi^2 + \Delta^2)^{\frac{1}{2}}} \tanh \frac{(\xi^2 + \Delta^2)^{\frac{1}{2}}}{2k_B T}, \tag{2}$$

may be written as

$$\ln \frac{\Delta_0}{\Delta} = 2 \int_0^{\hbar\omega_D} \frac{d\xi}{(\xi^2 + \Delta^2)^{\frac{1}{2}}} \frac{1}{1 + e^{\beta(\xi^2+\Delta^2)^{\frac{1}{2}}}}, \tag{3}$$

where Δ is the gap at temperature T and $\Delta_0 = \Delta(T = 0)$, and hence derive

$$\Delta(T) \approx \Delta_0 - (2\pi\, \Delta_0\, k_B T)^{\frac{1}{2}} e^{-\Delta_0/k_B T}, \quad T \ll T_c. \tag{4}$$

(b) Near T_c use

$$\Delta = \frac{g}{\beta\hbar} \sum_n \int \frac{d^3k}{(2\pi)^3} \frac{\hbar\Delta}{(\hbar\omega_n)^2 + E_k^2}, \tag{5}$$

to prove that

$$\ln \frac{T}{T_c} = -\frac{7\zeta(3)}{8} \left(\frac{\Delta}{\pi\, k_B T}\right)^2 + O\left(\frac{\Delta}{k_B T}\right)^4, \tag{6}$$

and hence prove

$$\begin{aligned}\Delta(T) &\approx k_B\, T_c\, \pi \left(\frac{8}{7\zeta(3)}\right)^{\frac{1}{2}} \left(1 - \frac{T}{T_c}\right)^{\frac{1}{2}} \\ &\approx 3.06\, k_B\, T_c \left(1 - \frac{T}{T_c}\right)^{\frac{1}{2}}, \quad T_c - T \ll T_c\end{aligned} \tag{7}$$