Exercise-sheet 5 (June 13, 2017)

1 In class exercise: Formal properties of Green's functions

2 Homework - due Date: June 20, 2017 (20 points).

2.1 Finite temperature BCS theory: revisiting Gorkov equations

Consider the (mean-field) BCS Hamiltonian

$$\begin{aligned} H &= \int d^3x \psi^\dagger_\alpha(\boldsymbol{x}) \left[\frac{1}{2m}\left(-i\hbar\boldsymbol{\nabla} + \frac{e}{c}\boldsymbol{A}(\boldsymbol{x})\right)^2 - \mu\right]\psi_\alpha(\boldsymbol{x}) \\ &\quad -g\int d^3x \left[\langle\psi^\dagger_\downarrow(\boldsymbol{x})\psi^\dagger_\uparrow(\boldsymbol{x})\rangle\psi_\uparrow(\boldsymbol{x})\psi_\downarrow(\boldsymbol{x}) + \psi^\dagger_\downarrow(\boldsymbol{x})\psi^\dagger_\uparrow(\boldsymbol{x})\langle\psi_\uparrow(\boldsymbol{x})\psi_\downarrow(\boldsymbol{x})\rangle\right], \end{aligned} \tag{1}$$

where $\boldsymbol{A}(\boldsymbol{x})$ is the vector potential and $-e$ is the electronic charge.

Eq. (1) holds for an attractive δ-function potential, valid within the static limit of the phonon's spectrum.

(a) Derive the equations of motion for the field operators $\psi_\uparrow(\boldsymbol{x}\tau)$ and $\psi^\dagger_\downarrow(\boldsymbol{x}\tau)$, where

$$\psi_\uparrow(\boldsymbol{x}\tau) = e^{H\tau/\hbar}\,\psi_\uparrow(\boldsymbol{x})e^{-H\tau/\hbar}, \tag{2}$$
$$\psi^\dagger_\downarrow(\boldsymbol{x}\tau) = e^{H\tau/\hbar}\,\psi^\dagger_\downarrow(\boldsymbol{x})e^{-H\tau/\hbar}, \tag{3}$$

with $\tau = it$ is complex.

(b) Show the single-particle (temperature) Green's function

$$G(\boldsymbol{x}\tau, \boldsymbol{x}'\tau') = -\langle T_\tau\psi_\uparrow(\boldsymbol{x}\tau)\psi^\dagger_\uparrow(\boldsymbol{x}'\tau')\rangle, \tag{4}$$

satisfies the equation of motion

$$\left[-\hbar\frac{\partial}{\partial\tau} - \frac{1}{2m}\left(-i\hbar\boldsymbol{\nabla} + \frac{e\boldsymbol{A}}{c}\right)^2 + \mu\right]G(\boldsymbol{x}\tau,\boldsymbol{x}'\tau') + \Delta(\boldsymbol{x})F^\dagger(\boldsymbol{x}\tau,\boldsymbol{x}'\tau') = \hbar\,\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(\tau-\tau'), \tag{5}$$

with

$$F^{\dagger}(\boldsymbol{x}\tau, \boldsymbol{x}'\tau') \equiv -\langle T_\tau \psi_\downarrow^\dagger(\boldsymbol{x}\tau)\psi_\uparrow^\dagger(\boldsymbol{x}'\tau')\rangle, \tag{6}$$

$$\Delta(\boldsymbol{x}) \equiv gF(\boldsymbol{x}\tau^+, \boldsymbol{x}\tau) = -g\langle \psi_\uparrow(\boldsymbol{x})\psi_\downarrow(\boldsymbol{x})\rangle = g\langle \psi_\downarrow(\boldsymbol{x})\psi_\uparrow(\boldsymbol{x})\rangle. \tag{7}$$

(c) Derive the analog equation of motion for $F^{\dagger}(\boldsymbol{x}\tau, \boldsymbol{x}'\tau')$.