Exercise-sheet 4 (May 30, 2017)

1 Homework - due Date: June 6, 2017 (30 points).

1.1 Matsubara Green's function (15 points)

The temperature or Matsubara Green's function is defined as follows

$$
\begin{aligned}
G(\tau,\tau') &= -\langle T_\tau \hat{A}(\tau)\hat{B}(\tau')\rangle \\
&= -\langle \hat{A}(\tau)\hat{B}(\tau')\rangle\theta(\tau-\tau') + \epsilon\langle \hat{B}(\tau')\hat{A}(\tau)\rangle\theta(\tau'-\tau),
\end{aligned}
\tag{1}
$$

where T_τ is the imaginary time-ordering operator, $\theta(\tau-\tau')$ is the Heaviside step function and τ is the imaginary time defined as $\tau = it$, with t the real (physical) time. The brackets $\langle\cdot\rangle$ are to be understood as *thermodynamic expectation values*. The imaginary time-dependent operators are defined as $\hat{X}(\tau) = e^{\tau\hat{H}}\hat{X}e^{-\tau\hat{H}}$ and $\hat{H}$ the Hamiltonian of the system under study. Finally, if $\hat{A}$ and $\hat{B}$ are bosonic operators ϵ is taken to be (-1), otherwise it is taken to be 1.

(a) Assume $\tau > \tau'$ and use the ciclic properties of the trace to show $G(\tau,\tau') = G(\tau-\tau',0)$.

(b) Consider $G(\tau) \equiv G(\tau,0)$ for $-\beta < \tau < 0$, with β the inverse temperature, and use the cyclic properties of the trace to show

$$G(\tau) = -\epsilon\, G(\tau+\beta). \tag{2}$$

Equation (2) implies that in the interval $-\beta < \tau < \beta$, $G(\tau)$ can be expanded in a Fourier series as follows

$$G(\tau) = \frac{1}{\beta}\sum_l \tilde{G}(\omega_l)e^{-i\omega_l\tau}, \tag{3}$$

with $\tilde{G}(\omega_l) = \displaystyle\int_0^\beta d\tau e^{i\omega_l\tau}G(\tau)$.

(c) What values can take ω_l so that equation (2) holds?

(d) Let's denote by $|m\rangle$ and E_m the exact eigenstates and eigenvalues of the Hamiltonian $\hat{H}$, i.e. $\hat{H}|m\rangle = E_m|m\rangle$. Show that the Fourier transformed Matsubara Green's function $\tilde{G}(\omega_l)$ can be written as

$$\tilde{G}(\omega_l) = -\frac{1}{Z}\sum_{n,m} A_{nm}\, B_{mn}\, \frac{e^{-\beta E_n} + \epsilon\, e^{-\beta E_m}}{E_m - E_n - i\omega_l}, \tag{4}$$

with Z the partition function, $A_{nm} = \langle n|\hat{A}|m\rangle$ and $B_{mn} = \langle m|\hat{B}|n\rangle$.
Since the energies E_m are real numbers, one can rewrite equation (4) as

$$\tilde{G}(\omega_l) = \int_{-\infty}^{\infty} dx\, \frac{A(x)}{i\omega_l - x}, \tag{5}$$

where

$$A(x) = \frac{(1 + \epsilon\, e^{-\beta x})}{Z} \sum_{n,m} e^{-\beta E_n} A_{nm} B_{mn} \delta(x - E_m + E_n), \tag{6}$$

is the spectral function, which was derived in the lecture.

(e) Integrate equation (6) over the entire real line and obtain the corresponding sum rule.

1.2 Finite momentum Cooper pairs (15 points)

Denote a Cooper pair with momentum $\boldsymbol{p}$ by

$$|\Psi(\boldsymbol{p})\rangle = \sum_{\boldsymbol{k}} g_{\boldsymbol{k}} |\boldsymbol{k}\boldsymbol{p}\rangle, \tag{7}$$

with $|\boldsymbol{k}\boldsymbol{p}\rangle = \hat{c}^\dagger_{\boldsymbol{k}+\boldsymbol{p}/2\uparrow}\hat{c}^\dagger_{-\boldsymbol{k}+\boldsymbol{p}/2\downarrow}|\psi_0\rangle$, where $|\psi_0\rangle$ labels the Fermi sea, and the $g_{\boldsymbol{k}}$ are Fourier coefficients.

(a) Show that if $|\Psi(\boldsymbol{p})\rangle$ is an eigenstate of

$$\hat{H} = \sum_{\boldsymbol{k},\sigma} \epsilon_{\boldsymbol{k}} \hat{c}^\dagger_{\boldsymbol{k}\sigma}\hat{c}_{\boldsymbol{k}\sigma} + \hat{V}, \tag{8}$$

with eigenvalue $E_{\boldsymbol{p}}$, where the matrix elements of the interaction $\langle \boldsymbol{k}\boldsymbol{p}|\hat{V}|\boldsymbol{k'}\boldsymbol{p}\rangle g_{\boldsymbol{k'}} = -\lambda_0/V$ is independent of $\boldsymbol{p}$, it follows that

$$g_{\boldsymbol{k}} = \frac{\lambda_0/V}{\epsilon_{\boldsymbol{k}+\boldsymbol{p}/2} + \epsilon_{\boldsymbol{k}-\boldsymbol{p}/2} - E_{\boldsymbol{p}}} \sum_{0<\epsilon_{\boldsymbol{k'}\pm\boldsymbol{p}/2}<\omega_D} g_{\boldsymbol{k'}}, \tag{9}$$

where ω_D is the Debye frequency and $\epsilon_{\boldsymbol{k}}$ is measured from the Fermi surface.

(b) Rewrite Eq. (9) as

$$1 - \frac{\lambda_0}{V} \sum_{0<\epsilon_{\boldsymbol{k}\pm\boldsymbol{p}/2}<\omega_D} \frac{1}{\epsilon_{\boldsymbol{k}+\boldsymbol{p}/2} + \epsilon_{\boldsymbol{k}-\boldsymbol{p}/2} - E_{\boldsymbol{p}}} = 0, \tag{10}$$

and perform the energy integration by, as usual, assuming a constant density of states $N(0)$ and imposing the condition

$$\epsilon_{\boldsymbol{k}\pm\boldsymbol{p}/2} > \frac{pv_F}{2}|\cos\theta| - \frac{p^2}{8m}, \tag{11}$$

to show that if $v_F p \ll |E_{\boldsymbol{p}}|$, with v_F the Fermi velocity, Eq. (10) simplifies to

$$1 - \frac{\lambda_0 N(0)}{2} \ln \frac{2\omega_D}{v_F p - E_{\boldsymbol{p}}} = 0. \tag{12}$$

(c) Solve Eq. (12) for $E_{\boldsymbol{p}}$.

The linear spectrum you have obtained is a signature of a *collective, bosonic mode.*