Exercise-sheet 2 (May 16, 2017)

1 Migdal's theorem

Migdal argued (*Sov. Phys. JETP*, **34**, 996 (1958)) that for electrons in normal metals the corrections to the leading term $g_{\boldsymbol{q}}$ in the vertex series are of the order $(m/M)^{1/2} g_{\boldsymbol{q}} \approx 10^{-2} g_{\boldsymbol{q}}$, i.e.

$$\Gamma = g\left\{1 + O\left[\left(\frac{m}{M}\right)^{1/2}\right]\right\}. \tag{1}$$

In this exercise session we shall not proof this claim but rather give arguments for its validity to first order, following mainly Fetter and Walecka.

2 Homework - due date: May 23, 2017 (20 + 5 points).

2.1 On the ground state of the Anderson model (10 points)

Consider the Anderson impurity model

$$H = \sum_{\sigma} \epsilon_f c^{\dagger}_{f\sigma} c_{f\sigma} + U n_{f\downarrow} n_{f\uparrow} + \sum_{\boldsymbol{k}\sigma} \epsilon_{\boldsymbol{k}} c^{\dagger}_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} + \sum_{\boldsymbol{k}\sigma} V_{\boldsymbol{k}f} (c^{\dagger}_{\boldsymbol{k}\sigma} c_{f\sigma} + c^{\dagger}_{f\sigma} c_{\boldsymbol{k}\sigma}), \tag{2}$$

which describes a magnetic impurity with orbital energy ϵ_f embedded in a metal. The corresponding creation (annihilation) operators are $c^{\dagger}_{f\sigma}$ ($c_{f\sigma}$) and n^f_σ the number operator. The Coulomb repulsion between the f electrons is denoted by U. On the other hand, the free conduction electrons in Bloch states are described by the operators $c^{\dagger}_{\boldsymbol{k}\sigma}$ ($c_{\boldsymbol{k}\sigma}$) and have dispersion $\epsilon_{\boldsymbol{k}}$, measured with respect to the Fermi energy ϵ_F. Their hybridization with the f electrons has strength $V_{\boldsymbol{k}f}$.

Varma and Yafet (*Phys. Rev. B*, **13**, 2950 (1975)) considered the variational wavefunction

$$|\Psi\rangle = a_0 |\Psi_0\rangle + \sum_{\boldsymbol{k}} a_{\boldsymbol{k}} (c^{\dagger}_{f\uparrow} c_{\boldsymbol{k}\uparrow} + c^{\dagger}_{f\downarrow} c_{\boldsymbol{k}\downarrow}) |\Psi_0\rangle, \tag{3}$$

where $|\Psi_0\rangle = \prod_{\boldsymbol{k}}^{k_F} c^{\dagger}_{\boldsymbol{k}\uparrow} c^{\dagger}_{\boldsymbol{k}\downarrow} |0\rangle$ is the ground state of the free metallic band, and the coefficients a_0 and $a_{\boldsymbol{k}}$ are complex variational parameters, fixed by the condition of minimum variational energy.

(a) Compute the variational energy

$$E = \langle\Psi|H|\Psi\rangle \equiv E_0 + \epsilon_f + \epsilon, \tag{4}$$

where E_0 is the unperturbed energy of $|\Psi_0\rangle$ and ϵ is to be determined.

(b) Show the variational equations for the coefficients a_0 and $a_{\boldsymbol{k}}$ read

$$\epsilon a_0 = -\epsilon_f a_0 + \sum_{\boldsymbol{k}} \sqrt{2} V_{\boldsymbol{k}f} a_{\boldsymbol{k}}, \tag{5}$$

$$\epsilon a_{\boldsymbol{k}} = \sqrt{2} V_{\boldsymbol{k}f} a_0 - \epsilon_{\boldsymbol{k}} a_{\boldsymbol{k}}. \tag{6}$$

(c) Use Eqs. (5) and (6) to write down the eigenvalue problem

$$\epsilon = -\epsilon_f + \sum_{\boldsymbol{k}} \frac{2V_{\boldsymbol{k}f}^2}{\epsilon + \epsilon_{\boldsymbol{k}}}. \tag{7}$$

Note that Eq. (7) has the exact same form as the eigenvalue problem derived in the context of the Cooper problem.

(d) Solve Eq. (7) by repeating the same arguments used in the lecture.

2.2 Into the gory details of Migdal's theorem proof (to first order) (10 points)

As a continuation to Migdal's theorem proof, evaluate the integrals

$$I_1 = \int \frac{d^4 l}{(2\pi)^4} \frac{\omega_{\boldsymbol{l}}^2}{l_0^2 - \omega_{\boldsymbol{l}}^2 + i\delta} \frac{\theta(\omega_D - \omega_{\boldsymbol{l}})}{(k_0 - l_0 - \epsilon_{\boldsymbol{k}-\boldsymbol{l}} + i\eta S_{\boldsymbol{k}-\boldsymbol{l}})(k_0 + q_0 - l_0 - \epsilon_{\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{l}} + i\eta S_{\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{l}})}, \tag{8}$$

and

$$I_2 = \int \frac{d\zeta}{\zeta - i\eta S_{k_0}} \ln \frac{\zeta - q_0 - q(\partial\epsilon_t/\partial t)_{\epsilon_t = k_0} - i\eta s_{k_0+q_0}}{\zeta - q_0 + q(\partial\epsilon_t/\partial t)_{\epsilon_t = k_0} - i\eta s_{k_0+q_0}}, \tag{9}$$

where $S_{\boldsymbol{q}} = 1(-1)$ if $|\boldsymbol{q}| > 1$ ($|\boldsymbol{q}| < 1$) and $s_{q_0} = 1(-1)$ if $q_0 > 1$ ($q_0 < 1$)

2.3 Bonus: first order perturbation theory (5 points)

Revisit the derivation of the zero-temperature perturbation theory (i.e. interaction picture, Wick's theorem, etc) in e.g. Fetter and Walecka or Mattuck, and write down the explicit series expansion to first order for a generic two-body interaction and the corresponding Feynman diagrams.